Definition 1. A *function* is an operator or a procedure which accepts a permissible input and transforms it into a unique output. The input is some nonempty set. If a function is defined to be

$$y = \mathrm{f(x)}$$

then x is the input vector, y the output, and $\mathrm{f}(\cdot)$ the function itself.

Example 1.

If f(·) is the function of dressing, then its input is possibly a person and its output a dressed person.

If f(·) is the function of making up, then the input is perhaps a girl and the output a made-up girl.

Definition 2. Let

$$y = \mathrm{f}(\mathrm{a}, \mathrm{x})$$

be a function, where a is a set of all its parameters, and x a set of all its variables. Then y is its dependent variable and x_i, for all $i \in \mathrm{x}$, are its independent variables.

In other words, x_i vary, y follows, and a_i could assume any value within the range of its permissible ones, but its value must be constant.

Definition 3. An *inverse function* is an expression of the independent variable in terms of the dependent variables. The inverse of the function $\mathrm{f}(\cdot)$ is denoted by $\mathrm{f}^{-1}(\cdot)$. If $\mathrm{f}(\cdot)$ is a function which admits one independent variable, namely x, then one could express it as,

$$y = \mathrm{f}(x) \tag{1}$$

Its inverse function is then,

$$\mathrm{f}^{-1}(y) = x \tag{2}$$

Example 2. Both the function and its inverse may be thought of as being an operator operating on an input to produce an output. The function,

$$y = \mathrm{f}(x)$$

is understood diagrammatically as,

$$y \leftarrow \boxed{\mathrm{f}(\cdot)} \leftarrow x$$

while its inverse function,

$$x = \mathrm{f}^{-1}(y)$$

is displayed as a diagram as,

$$y \rightarrow \boxed{\mathrm{f}^{-1}(\cdot)} \rightarrow x$$

Theorem 1. An inverse function must always be a one-to-one mapping.

Proof. Let $f(\cdot)$ be a function. Then $f(\cdot)$ can be either one-to-one or many-to-one, and therefore $f^{-1}(\cdot)$ could turn out to be either one-to-one or one-to-many. But since f^{-1} is also a function, so for each of the values in its domain the corresponding value in its range must be unique. This means that in cases where f^{-1} turns out to be one-to-many, some constraints must be put on its input in order to make the output one-to-one, which then makes all the outputs from $f^{-1}(\cdot)$ one-to-one. ¶

$f(\cdot)$	$f^{-1}(\cdot)$
addition	subtraction
multiplication	division
power	root
exponential	logarithm

Table 1 *Some of the functions and their corresponding inverse functions.*

$\mathrm{f}(\cdot)$	$\mathrm{f}^{-1}(\cdot)$
$x + a$	$x - a$
$x \cdot a$	$\frac{x}{a}$
x^a	$\sqrt[a]{x}$
a^x	$\log_a x$

Table 2 *The notational forms of functions and their inverses.*

in which division and logarithm are both undefined for $a = 0$.

Definition 4. The inverse of the addition,

$$y = x + a$$

is the subtraction,

$$y - a = x$$

The inverse of the multiplication,

$$y = ax$$

is the division,

$$\frac{y}{a} = x$$

The inverse of the power,

$$y = x^a$$

is the root,

$$\sqrt[a]{y} = x$$

The inverse of the exponential,

$$y = a^x$$

is the logarithm,

$$\log_a y = x$$

Example 3.

$$\begin{array}{ccc} \underbrace{x \cdots x}_{b} & \rightarrow & x^b \\ \uparrow & & \\ x \cdot x = x^2 & & \\ \uparrow & & \\ \underbrace{x + \ldots + x}_{a} & \rightarrow & x \cdot a \\ \uparrow & & \\ x + x = 2x & & \end{array}$$

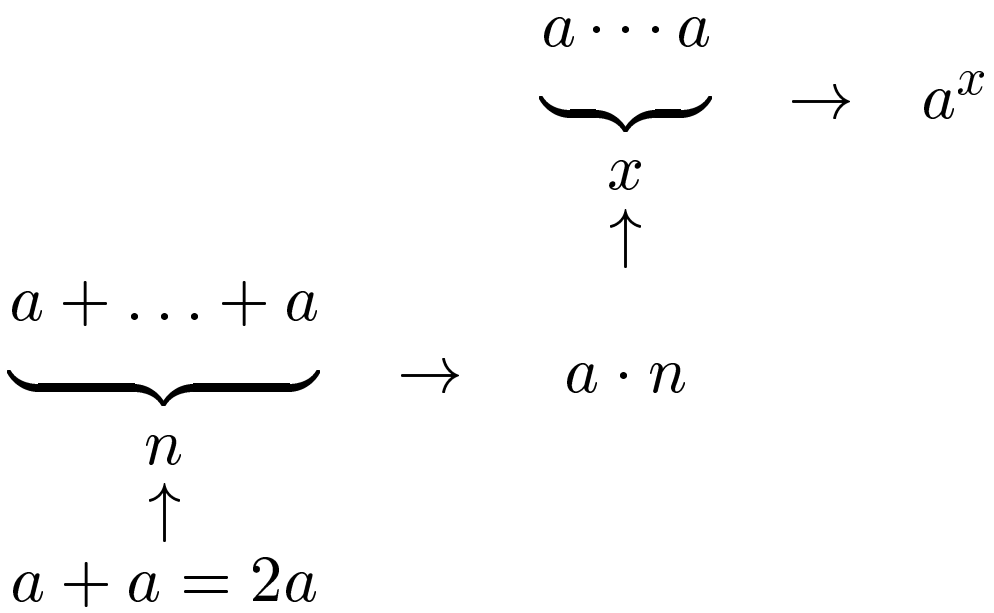

Figure 2 *Starting from a constant to obtain in the end the exponential function.*

Definition 5. An *exponential function* is defined as

$$y = a^x$$

where $a > 0$ and $a \neq 1$.

Example 4. The domain of the exponential function $y = a^x$ is the set of all real numbers, while its range the set of all positive real numbers. The function is convex and increasing when $a > 1$, and convex and decreasing when $0 < a < 1$. At $x = 0$, the value of the function is $y = 1$ for any $a > 0$.

Theorem 2. For any $a \neq 0$,

$$\lim_{x \to 0} a^x = 1$$

Theorem 3. Three basic rules of the exponential function are,

$$a^m a^n = a^{m+n}$$
$$\frac{a^m}{a^n} = a^{m-n}$$
$$(a^m)^n = a^{mn}$$

Proof. Write, say, a^m as,

$$\underbrace{a \cdots a}_{m}$$

and similarly for a^n. Then all three equations above become obvious. ¶

Example 5.

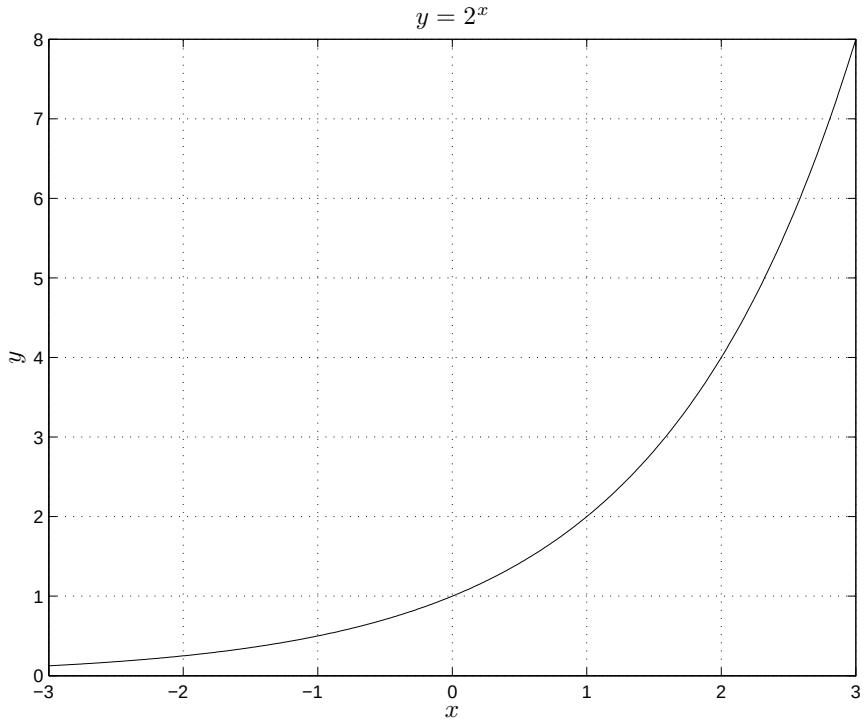

Figure 3 *Example of the graph of the exponential function when $a > 1$. Here the graph is that of $y = 2^x$.*

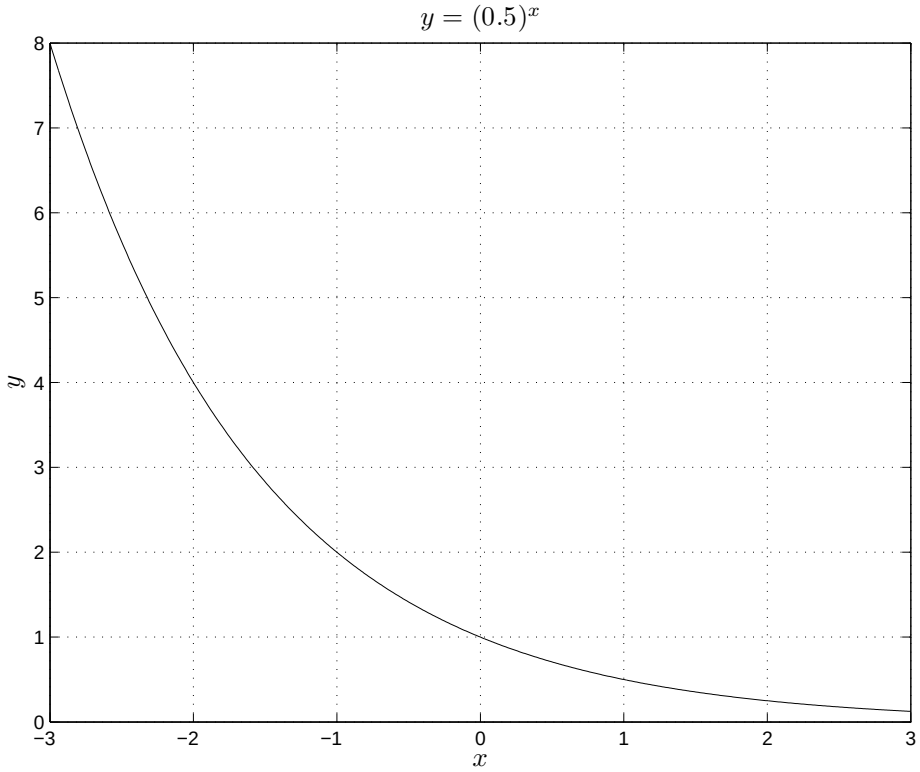

Figure 4 *An example of graph of the exponential function* $y = a^x$ *when* $0 < a < 1$. *Here* $a = 0.5$.

Definition 6. Let $a > 1$. Then the graph of

$$y = a^x$$

is called a *growth curve*, while that of

$$y = a^{-x}$$

is called a *decay curve*.

Example 6. There are basically three laws of growth, namely unlimited, limited and logistic growth, all of which involve an exponential function. The model is for unlimited growth,

$$y(t) = ae^{rt}$$

for limited growth,

$$y(t) = m\left(1 - e^{-rt}\right)$$

and for logistic growth,

$$y(t) = \frac{m}{1 + ae^{-rmt}}$$

where a, m and r are constants.

Example 7. The value of a principal p compounded annually at an interest rate i for t years is,

$$s = p(1+i)^t$$

where i is expressed in decimal points. For compounding m times a year, then,

$$s = p\left(1+\frac{i}{m}\right)^{mt}$$

If the compounding is continuous, at 100 per cent interest for one year, then,

$$s = p\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m = pe$$

where e is the Euler's constant, $e = 2.71828\ldots$

Example 8. For multiple compounding,

$$p(1+i_e)^t = p\left(1+\frac{i}{m}\right)^{mt}$$

the effective annual rate of interest is,

$$i_e = \left(1+\frac{i}{m}\right)^m - 1$$

The effective annual rate of interest for continuous compounding is,

$$i_e = e^r - 1$$

Definition 7. *Discounting* is the process of finding the present value p of a future sum of money s.

Example 9. Discounting when under annual compounding is,

$$s = p(1+i)^t$$

when under multiple compounding,

$$p = s\left(1+\frac{i}{m}\right)^{-mt}$$

and when under continuous compounding,

$$p = se^{-rt}$$

When discounting, the interest rate i is called the *rate of discount.*

Example 10. A discrete growth $s = p(1 + i/m)^{mt}$ can be converted to a continuous growth $s = pe^{rt}$ thus,

$$p\left(1+\frac{i}{m}\right)^{mt} = pe^{rt}$$

$$\ln\left(1+\frac{i}{m}\right)^{mt} = \ln e^{rt}$$

$$r = m\ln\left(1+\frac{i}{m}\right)$$

Therefore,

$$s = p\left(1+\frac{i}{m}\right)^{mt} = pe^{m\ln\left(1+\frac{i}{m}\right)t}$$

Example 11. Reversing the sign of x, that is replacing x by $-x$, has the effect of reflection of the original graph with respect to the y-axis. Reversing the sign of y, that is replacing y by $-y$, gives a reflection of the same with respect to the x-axis. The graphs of $y = a^{\pm x}$ remain always above the x-axis, in other words the function $y = a^{\pm x}$ maps $-\infty < x < \infty$ to $y > 0$. The two functions $y = a^x$ and $y = a^{-x}$ are the reflection of each other with respect to the y-axis. It can be easily seen that the functions $y = -a^{\pm x}$ are the reflection with respect to the x-axis respectively of $y = a^{\pm x}$.

Definition 8. The *logarithmic function* with base a is defined to be the inverse of the exponential function, and is written

$$y = \log_a x$$

where $a > 0$ and $a \neq 1$. The logarithmic function of base 10 is called the *common logarithmic function*, and one of base e, where

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

is called the *natural logarithmic function*. By the notation $y = \log_a x$ we mean that the logarithm base a of x is the power to which a must be raised to get x.

Example 12. The domain of the logarithmic function

$$y = \log_a x$$

is the set of all positive real numbers, its range the set of all real numbers.

The function is concave and increasing for $a > 1$, and is convex and decreasing for $0 < a < 1$. Note also that $\log_a x$ is the power which a must be raised to get x.

Example 13. Note that

$$e^{\ln a} = a = \ln e^{a}$$

where $a > 0$,

$$e^{\ln x} = x = \ln e^{x}$$

where $x > 0$, and

$$e^{\ln \mathrm{f}(x)} = \mathrm{f}(x) = \ln e^{\mathrm{f}(x)}$$

where $\mathrm{f}(x) > 0$.

Theorem 4. Four basic rules for logarithm function are listed in the following.

$$\log_b m + \log_b n = \log_b mn$$

$$\log_b m - \log_b n = \log_b \frac{m}{n}$$

$$\log_b m^z = z \log_b m$$

$$\log_b n = \frac{\log_x n}{\log_x b}$$

Definition 9. The *elasticity of substitution* σ is defined as,

$$\sigma = \frac{\frac{\mathrm{d}\left(\frac{k}{l}\right)}{\frac{k}{l}}}{\frac{\mathrm{d}\left(\frac{p_l}{p_k}\right)}{\frac{p_l}{p_k}}} = \frac{\frac{\mathrm{d}\left(\frac{k}{l}\right)}{\mathrm{d}\left(\frac{p_l}{p_k}\right)}}{\frac{\frac{k}{l}}{\frac{p_l}{p_k}}}$$

where $\frac{k}{l}$ is called the *least-cost input ratio*, and $\frac{p_l}{p_k}$ the *input-price ratio*.

Example 14. The value $\sigma = 0$ means there is no substitutability, that is the two inputs are complements of each other and both must be used together in a fixed proportion.

The value $\sigma = \infty$ means that the two goods may substitute each other perfectly. Ultimately, $0 \leq \sigma \leq \infty$.

Definition 10. A *constant elasticity of substitution production function* is a production function where, unlike the Cobb-Douglas function, has an elasticity of substitution whose value is constant but not necessarily 1. In its typical form, it is,

$$q = a\left(\alpha k^{-\beta} + (1-\alpha)l^{-\beta}\right)^{-\frac{1}{\beta}}$$

where a is called the *efficiency parameter*, α the *distribution parameter*, β the *substitution parameter*. Furthermore, β determines σ, and $a > 0$, $0 < \alpha < 1$, and $\beta > -1$.

Example 15. Some nonlinear functions can be converted to linear functions using logarithmic transformation, for example the Cobb-Douglas production function,

$$q = ak^{\alpha}l^{\beta}$$

which becomes

$$\ln q = \ln a + \alpha \ln k + \beta \ln l$$

Other nonlinear functions can not be converted, for example the constant elasticity of substitution production function,

$$q = a\left[\alpha k^{-\beta} + (1-\alpha)l^{-\beta}\right]^{-\frac{1}{\beta}}$$

which becomes just another nonlinear function,

$$\ln q = \ln a - \frac{1}{\beta}\ln\left[\alpha k^{-\beta} + (1-\alpha)l^{-\beta}\right]$$

others

Example 16. Let the total revenue be

$$r_t = pq$$

and the demand function

$$p = a - bq$$

where q is the quantity sold. Then r_t expressed as a function of q is nonlinear, for

$$r_t = (a - bq)q = aq - bq^2$$

Example 17. A more realistic equation for the total cost instead of

$$c_t = a + bq$$

is the nonlinear function

$$c_t = aq^3 - bq^2 + cq + d$$

in which the production cost increases with quantity in at a decreasing rate ($c_t' < 0$) up to the inflection point at

$$q = \frac{b}{6a}$$

after which it increases at an increasing rate ($c_t'' > 0$).

Definition 11. A *polynomial* is an expression in the form

$$\sum_{i=0}^{n} a_i x^{n-i}$$

Here n is called the *order* of the polynomial. If $n = 2$ the polynomial is known as a *quadratic polynomial*, if $n = 3$ a *cubic polynomial*, if $n = 4$ a *quartic*, $n = 5$ a *quintic* and $n = 6$ a *sextic*. If we let $\mathrm{p}(x)$ be a polynomial, then a *polynomial equation* is the equation $\mathrm{p}(x) = 0$. A *polynomial function* is a function of the form

$$y = \mathrm{f}(x) = \mathrm{p}(x)$$

Example 18. The quadratic equation $ax^2 + bx + c = 0$ has the solutions,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{3}$$

These solutions are called the *roots* of the quadratic equation. Equation 3 is called the 'minus-b formula'. The values of x obtained from the minus-b formula give the intersections of the graph of the quadratic function

$$\mathrm{f}(x) = \mathrm{p}(x) = ax^2 + bx + c$$

on the x-axis.

The value $b^2 - 4ac$ determines how the graph of $\mathrm{f}(x)$ lies relative to the x-axis, that is,

$$b^2 - 4ac \begin{cases} > 0, & \text{there are two } x\text{-intersections} \\ = 0, & \text{the graph touches the } x\text{-axis at one point} \\ < 0, & \text{the graph never touches the } x\text{-axis} \end{cases}$$

Furthermore, the graph reverses its direction with respect to the y-axis at the critical point where $\mathrm{f}'(x) = 0$, that is when

$$x = -\frac{b}{2a}$$

Consequently the critical point is

$$\left(-\frac{b}{2a}, -\frac{b^2}{4a} + c\right)$$

The graph of f(x) is symmetric with respect to the vertical line which passes through the turning point, that is to say, the line

$$x = -\frac{b}{2a}$$

The y-intercept is at the point $(0, c)$.

Example 19. A hyperbolic relation has the form,

$$(px+q)(ry+s)=t$$

From this we obtain,

$$\left(x+\frac{q}{p}\right)\left(y+\frac{s}{r}\right)=\frac{t}{pr}$$

$$y=\frac{t}{pr}\left(\frac{1}{x+\frac{q}{p}}\right)-\frac{s}{r}$$

$$=\frac{a}{x+b}-c$$

where p, q, r, s and t are constants, hence so are $a=\frac{t}{pr}$, $b=\frac{q}{p}$, $c=\frac{s}{r}$.

In economics we sometimes find hyperbolic functions of the form,

$$y = \frac{a}{bx + c} \tag{4}$$

For example, a demand function of a good may be given by,

$$q + a = \frac{m}{p}$$

which leads to,

$$p = \frac{m}{q + a}$$

where p and q are respectively price and quantity demanded of a good, while m and a are constants.

The graph of Equation 4 has the x-axis, that is the line $y = 0$, as its horizontal asymptote, and has the line

$$x = -\frac{c}{b}$$

as its vertical asymptote. If all the parameters are positive, then the curve in the first quadrant decreases with a decreasing rate.